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Journal of Geometry and Physics 25 (1998) 341–345

JOURNAL OF GEOMETRY AND PHYSICS

Brief Communication

On the Wigner angle and its relation with the defect of a triangle in hyperbolic geometry ★

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Received 8 July 1997; received in revised form 6 September 1997

Abstract

We present a theorem on the Wigner angle and its relation with the defect of a triangle in hyperbolic geometry. Copyright © 1998 Elsevier Science B.V.

Subj. Class.: Quantum mechanics; General relativity
1991 MSC: 70H40, 83A05
Keywords: Wigner angle; Defect of a triangle; Hyperbolic geometry

1. Main result and Wigner angle

A reference frame in special relativity may be regarded as a point on a hyperboloid $\mathcal{H} := \{x^\mu x_\mu = -1; x^0 > 0\}$ (we have chosen the signature of the metric as $(-, +, +, +)$). Suppose we have three reference frames $A, B, C \in \mathcal{H}$. They define uniquely three boost transformations which will be denoted by L_{BA}, L_{CB} and L_{CA}. The transformation L_{BA} e.g. is defined as the unique boost such that $L_{BA}A = B$, etc.

Theorem 1. *The superposition $L_{AC}L_{CB}L_{BA} = R_\delta$ is equal to the three-dimensional rotation in the hyperplane orthogonal to A, around the axis $A \wedge B \wedge C$ and the angle of rotation δ is equal to the defect of the triangle $\{A, B, C\}$ with respect to the hyperbolic (Lobaczewski) geometry of $\mathcal{H}$.*

★ Supported in part by the National Natural Science Foundation of China.

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PII S0393-0440(97)00071-5

The three-dimensional rotation R_δ arising in Theorem 1 is the well-known *Wigner rotation* of special relativity, which gives rise to the so-called *Thomas precession*, a relativistic effect first encountered in the spin–orbit coupling of an electron in an atom [1–3]. The angle involved in the Wigner rotation is called the *Wigner angle*. It has been calculated by different authors by means of different mathematical techniques, such as 2×2 matrices (quaternions) (see [4]), Clifford algebra of differential forms (see [5]) and implicit Lorentz transformations (see [6]), and so on. The explicit vector form of the Wigner angle is

$$\delta = 2 \arcsin\left(\frac{\gamma_1\gamma_2 \mid \mathbf{w} \times \mathbf{v} \mid}{c^2\sqrt{2(1+\gamma_1)(1+\gamma_2)(1+\gamma)}}\right) \frac{\mathbf{w} \times \mathbf{v}}{\mid \mathbf{w} \times \mathbf{v} \mid}, \tag{1}$$

where $\mathbf{v}$ is the three-dimensional velocity of frame B relative to frame A, $\mathbf{w}$ is that of frame C relative to frame B, $\mathbf{w} \times \mathbf{v}$ indicates the axis of the rotation, c is the speed of light in empty space, and

$$\gamma_1 = 1\Big/\sqrt{1-\frac{v^2}{c^2}}, \quad \gamma_2 = 1\Big/\sqrt{1-\frac{w^2}{c^2}}, \quad \gamma = \gamma_1\gamma_2\left(1+\frac{\mathbf{v}\cdot\mathbf{w}}{c^2}\right). \tag{2}$$

From (1) we find

$$\cos\delta = 1 - \frac{(\gamma_1-1)(\gamma_2-1)}{1+\gamma}\sin^2\theta, \tag{3}$$

where

$$\sin\theta = \frac{\mathbf{v}\times\mathbf{w}}{\mid\mathbf{v}\mid\mid\mathbf{w}\mid}. \tag{4}$$

In the next section we shall point out the geometric meaning of the Wigner angle.

2. The defect of a triangle and its relation with the Wigner angle

Let

(i) $\mathbf{v}$ and $\mathbf{u}$ be the velocities of B and C relative to the reference frame A, respectively, and θ_A the angle between them measured in A.

(ii) $\mathbf{w}$ and $\mathbf{v}'$ be the velocities of C and A relative to the reference frame B, respectively, and θ_B the angle between them measured in B.

(iii) $\mathbf{u}'$ and $\mathbf{w}'$ be the velocities of A and B relative to the reference frame C, respectively, and θ_B the angle between them measured in C.

Denote by

$$\alpha = \theta_A + \theta_B + \theta_C, \tag{5}$$

the angle-sum of the three angles mentioned above.

Define

$$\delta = \pi - \alpha = \pi - (\theta_A + \theta_B + \theta_C), \tag{6}$$

where π is the ratio between the circumference of a circle and its diameter.

We ask: What is the magnitude of δ ?

Suppose the vectors in conditions (i)–(iii) satisfy the following relations:

$$|\mathbf{v}'| = |\mathbf{v}| = v, \qquad |\mathbf{w}'| = |\mathbf{w}| = w, \qquad |\mathbf{u}'| = |\mathbf{u}| = u. \tag{7}$$

If the physical world is interpreted by the theory of special relativity, then v, w and u satisfying

$$u = \frac{\sqrt{(v^2 + w^2 - 2vw\cos\theta_B) - (vw\sin\theta_B/c)^2}}{1 - vw\cos\theta_B/c^2} \tag{8}$$

can be written as [4]

$$\sqrt{1-\beta_u^2} = \frac{\sqrt{1-\beta_v^2}\sqrt{1-\beta_w^2}}{1-\beta_v\beta_w\cos\theta_B}, \tag{9}$$

or

$$\gamma_u = \gamma_v\gamma_w(1-\beta_v\beta_w\cos\theta_B), \tag{10}$$

where $\beta_u = u/c$, $\gamma_u = 1/\sqrt{1-\beta_u^2}$, and analogously for β_v, γ_v and so on.

One can easily verify that (10) coincides with the third equation of (2). From (10) we obtain

$$\cos\theta_B = \frac{\gamma_v\gamma_w - \gamma_u}{\gamma_v\gamma_w\beta_v\beta_w} \tag{11}$$

so that

$$\sin\theta_B = \sqrt{1-\cos^2\theta_B} = \frac{\sqrt{1-\gamma_v^2-\gamma_w^2-\gamma_u^2+2\gamma_v\gamma_w\gamma_u}}{\sqrt{\gamma_v^2-1}\sqrt{\gamma_w^2-1}}. \tag{12}$$

Obviously, due to relations (7), $\cos\theta_A$, $\sin\theta_A$, $\cos\theta_C$ and $\sin\theta_C$ can be obtained by replacing the subscripts involved in (11) and (12); from these we have

$$\cos\delta = 1 - \frac{(\gamma_v-1)(\gamma_w-1)}{1+\gamma_u}\sin^2\theta_B > 0 \tag{13}$$

and

$$\sin\delta = \frac{(1+\gamma_v+\gamma_w+\gamma_u)\sqrt{1-\gamma_v^2-\gamma_w^2-\gamma_u^2+2\gamma_v\gamma_w\gamma_u}}{(1+\gamma_v)(1+\gamma_w)(1+\gamma_u)} > 0. \tag{14}$$

As a result of (13) and (14), one can find

$$\delta > 0, \tag{15}$$

thus

$$\alpha = \pi - \delta < \pi. \tag{16}$$

Eq. (16) implies that the angle-sum α is less than π.

In hyperbolic geometry it is well known that the angle-sum of a triangle is less than π. It has been pointed out that the facts and concepts of relativistic mechanics could be thought of not only in terms of Minkowskian geometry (according to the original interpretation of Einstein's special theory of relativity), but also in terms of hyperbolic geometry [7]. The similarities between them enabled us to translate every theorem of hyperbolic geometry into a theorem of relativistic kinematics, and conversely it could be shown that, from the point of view of Einstein's theory of relativity, the set of uniform motions in space could be regarded as a hyperbolic space, provided the "velocity parameter" v is replaced by the "hyperbolic parameter" ϕ. Thus, we have

$$\tanh\phi = \frac{v}{c}, \qquad \cosh\phi = \gamma, \qquad \sinh\phi = \frac{v}{c}\gamma \tag{17}$$

If we define

$$\phi_A = \tanh^{-1}(w/c), \quad \phi_B = \tanh^{-1}(u/c), \quad \phi_C = \tanh^{-1}(v/c), \tag{18}$$

then due to (17), Eq. (10) can be written in a hyperbolic form

$$\cosh\phi_B = \cosh\phi_C \cosh\phi_A - \sinh\phi_C \sinh\phi_A \cos\theta_B. \tag{19}$$

It is clear that Eq. (19) is nothing but the *hyperbolic law of cosines*; to a first approximation it reduces to the law of cosines in Euclidean geometry [8]. Obviously it corresponds to the *theorem of addition of velocities* in special relativity. To see it clearly, we would like to consider the special case with $\theta_B = 0$; then (19) becomes

$$\cosh\phi_B = \cosh(\phi_A + \phi_C) \tag{20}$$

so that

$$\tanh\phi_B = \tanh(\phi_A + \phi_C) = \frac{\tanh\phi_A + \tanh\phi_C}{1 + \tanh\phi_A \tanh\phi_C}. \tag{21}$$

From (17), we obtain

$$u = \frac{v + w}{1 + vw/c^2} \tag{22}$$

this is just the relativistic addition law of two collinear vectors.

From this analysis, it is confirmed the belief that concepts arising in special relativity can have their correspondent in hyperbolic geometry. From the point of view of hyperbolic geometry, the hyperbolic parameters $\phi_i (i = A, B, C)$, as defined in (18), can be used to construct a triangle $\triangle ABC$, whose defect δ satisfies Eq. (13). One can verify that (13) is identical to (3), in which the Wigner angle is satisfied. Consequently the geometric meaning of the Wigner angle is nothing else than the defect of a triangle from the point of view of hyperbolic geometry. Thus Theorem 1 is proved.

Acknowledgements

One of the authors (J.L. Chen) is grateful to Dr. G.H. Chen, Prof.Y.Z. Ren and Prof.X.Q. Li for their valuable discussions.

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